CANTOR'S BIJECTION AND THE QUANTITY OF NUMBERS
(A BIJEÇÃO DE CANTOR E A QUANTIDADE DE NÚMEROS)

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ABSTRACT

There exists a paradox in mathematics when contrasting Cantor's one-to-one correspondence—which allows for a bijective function to be established between an infinite set and its proper subsets (as in the one-to-one mapping between the set of even numbers and the set of natural numbers)—with Euclid's principle that the part is less than the whole (the even numbers constitute a proper subset of the natural numbers and are thus "smaller"). In defense of Euclid's principle, we critique the use of bijective correspondence as a means of measuring the sizes of numerical sets and propose an alternative method for calculating these sizes. This new approach treats the position occupied by a number within the ordering represented by the real number line as an inherent aspect of the concept of number itself.

Keywords Cantor · Bijection · Cardinality · Sizes · Sets

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Conflitos de Interesse

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1 Introduction

The definition of what constitutes a number is problematic. This is widely acknowledged, as evidenced in [3]. However, this book mentions that a broadly accepted definition is John von Neumann's elegant recursive formulation: "every number is the set of smaller numbers" ([3, 12]).

As noted in [7], "The first number is the empty set, the next the set whose only element is the empty set, and so on". Various other definitions are possible, and in this latter work, I presented one such algorithmic-based definition.

The situation becomes even more complex when attempting to define how to quantify numbers, particularly concerning infinite numerical sets such as $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. Cantor was among those who employed bijective correspondence $\{\exists f : A \to B$ (bijective), where f is a function and A and B are the sets to be compared$\}$, to define the "cardinality"

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of a set, which can be used as a measure of quantity for numerical sets. Essentially, two sets S_1 and S_2 have the same cardinality, denoted $S_1 \sim S_2$, if a bijective correspondence can be established between their elements.

As discussed in [7], this definition leads to a paradox:

> Paolo [5] states, based on Cantor and Hume, that two countably infinite sets that can be placed in one-to-one correspondence have the same "size", that is, the same cardinality. However, this contradicts the principle, dating back to Euclid, that the part is less than the whole. A classic example of this contradiction is the comparison between the set of natural numbers and its subset of even numbers. They can be placed in one-to-one correspondence, for instance, via the function that maps each natural number n to the even number $2n$, which implies they have the same "size" under the Cantor-Hume principle. Yet, the even numbers, being a proper subset of the natural numbers, should have a smaller "size" according to Euclid's principle.

Gödel's assertion (see [7]) is also well-known:

> "Cantor's theory of infinite set size is inevitable", since "the number of objects belonging to some class does not change if, while leaving the objects unchanged, one alters their properties or mutual relations (e.g., their colors or spatial distribution)" [5, 637]. Clearly, Gödel here refers to the bijective correspondence between elements of two infinite sets.

As evident, there are fundamental issues with using Cantor's bijective correspondence to define a notion of quantity for infinite numerical sets. In this work, therefore, I present a critique, grounded in both philosophical and mathematical reasoning, regarding the validity of employing bijective correspondence as a measure of numerical quantity.

Before delving into this critique, it should be noted that a literature review and methodological exposition can be found in [7, 8, 9, 10].

2 Results and Discussion

Let us first consider the foundational concept used for defining the quantity of elements in sets: the bijective function. Two sets S_i and S_o are said to have the same quantity of elements if there exists a bijective function f between their elements, i.e., $\{\, \exists\, f : S_i \to S_o$ (bijective) $\}$.

The fundamental requirement of any bijective function, including this case, is that it "maps" each element $s_i \in S_i$ to a unique element $s_o \in S_o$. Considering the singleton sets $\{s_i\}$ and $\{s_o\}$, we can trivially state that the bijective function maps $\{s_i\} \to \{s_o\}$, indicating both sets have the same quantity of elements - one unit in this case.

More generally, assuming the existence and validity of bijective mappings between singleton sets, we can recursively construct (if it exists) a bijective function f between arbitrary sets S_i and S_o as follows: select unique elements $s_i \in S_i$ and $s_o \in S_o$, define $f(s_i) = s_o$, and recursively apply the same procedure to the reduced sets $S_i \setminus \{s_i\}$ and $S_o \setminus \{s_o\}$.

By mathematical induction, we can prove the following theorem:

Theorem 1. *For any two sets S_i and S_o, a bijective function f exists between their elements (i.e., $\{\, \exists\, f : S_i \to S_o$ (bijective) $\}$) if and only if there exists a bijective function between the elements of $S_i \setminus \{s_i\}$ and $S_o \setminus \{s_o\}$, where $s_i \in S_i$ and $s_o \in S_o$ are unique elements of their respective sets.*

Proof. Trivially, we acknowledge that a bijective function exists between any two empty sets - all empty sets have the same size. For non-empty sets S_i and S_o with unique elements s_i and s_o respectively, we proceed by induction: assuming the theorem holds for all proper subsets $S_{ii} \subsetneq S_i$ and $S_{oo} \subsetneq S_o$ (specifically when $S_{ii} = S_i \setminus \{s_i\}$ and $S_{oo} = S_o \setminus \{s_o\}$), the theorem remains valid when we add one unique element to each of these subsets and map them to each other. $\square$

For this definition of set cardinality to be meaningful, it must fundamentally rely on our ability to establish bijective correspondences between singleton subsets of arbitrary sets. Since these unique elements may be of entirely different natures, the crucial requirement for bijective correspondence is that all singleton sets have equal cardinality - trivially one unit. Any differences between elements must pertain to characteristics irrelevant for quantity measurement, such as color.

This definition inherently depends on the primitive concept of unit quantity. We can generally define the equivalence relation $S_i \sim S_o$ as relating two sets if and only if they have the same quantity of elements, without assumptions about the nature or characteristics of the elements beyond those intrinsically related to quantity.

This leads us to the fundamental question: what is quantity? Across sciences and philosophy, we encounter the distinction between discrete quantities (e.g., number of polygon sides) and continuous quantities (e.g., lengths of line segments). However, if we wish to understand quantity as a primitive concept preceding number, how should it be defined?

Without delving into the extensive philosophical debate on this matter, I maintain that quantity represents a fundamental ontological and mathematical concept distinct from number. Nevertheless, these concepts are intrinsically related: numbers can be viewed as labels associated with quantities. For instance, the natural number 2 refers to a specific quantity, with "2" serving merely as its label.

The concept of quantity inherently establishes an ordering, as we intuitively perceive greater and smaller quantities. This allows the representing numbers to be ordered in an increasing sequence starting from 0 (representing the null quantity of the empty set), followed by 1 (representing the unit quantity), and so forth through all natural numbers. Although constructing all numerical sets falls beyond this work's scope, we observe that this intrinsic ordering extends to continuous quantities and their numerical representations.

Real numbers (at least non-negative ones) can thus be viewed as labels intrinsically associated with both a quantity and a position in the total ordering of numbers. This establishes the well-known bijective correspondence between points on the non-negative real line and non-negative real numbers.

The crux of my argument is that the concept of real numbers cannot disregard their quantitative and ordinal attributes. When considering numbers, we must necessarily imagine both their associated quantity and their position in the total ordering. Stripped of these attributes, numbers become mere labels - no longer truly numbers.

If we disregard the ordering of numbers (or position in the non-negative real line), the conceptual framework creates significant difficulties when attempting to define the quantity of numbers in numerical sets. Each non-negative real number essentially represents an abstract quantity, so defining the quantity of numbers requires defining the quantity of quantities - a clear circularity.

The situation becomes particularly problematic when establishing bijective correspondences between sets of natural numbers (e.g., $\mathbb{N}$ and its even subset) while disregarding their quantitative and ordinal attributes. Such correspondences effectively compare mere arbitrary symbols. From this perspective, establishing bijections between infinite sequences of natural numbers differs little from creating correspondences between arbitrary symbol sequences (e.g., $\{*, \&, \%, \#, \dots\}$ and $\{+, @, x, !, \dots\}$).

The bijective correspondence between sets ultimately reduces to comparing labels stripped of quantitative meaning, only proving that both sets contain the same quantity of arbitrary symbols. When we consider real numbers with their full quantitative and ordinal attributes, they should be understood as ordered triples: (label, quantity) $\rightarrow$ (position on the real line).

Disregarding these attributes allows correct establishment of bijections between $\mathbb{N}$ and its proper subsets (e.g., evens), seemingly violating Euclid's principle that the whole exceeds its part (see [7]). This violation occurs precisely because treating numbers as mere symbols loses essential information about their quantitative ordering. For instance, the bijection $n \mapsto 2n$ between $\mathbb{N}$ and its even subset correctly maps $\{0, 1\}$ to $\{0, 2\}$, but "skips" the number 1 in the image set, ignoring that between 0 and 2 there should be three integers ($\{0, 1, 2\}$) but only two evens ($\{0, 2\}$). In general terms, for bijective correspondences between the set of natural numbers and any of its subsets, when we disregard the essential attributes of quantity and position on the real number line in this manner—"skipping" numbers that should properly be considered—we effectively discard the entire ordered structure of the number system. Consequently, we can establish bijective mappings between any two numerical sets that have thereby been reduced to mere arbitrary symbols.

Fundamentally, all bijective correspondences between arbitrary sets depend on functions mapping singleton sets. Consider $S_i = \{\#, *\} \rightarrow S_o = \{1, 2\}$ with $\# \mapsto 1$ and $* \mapsto 2$. If we consider 1 and 2 as mere labels, these mappings only confirm that both sets have equal cardinality (two units). Defining a quantity function Q over these symbols (if we wish to designate the operation of function Q with a specific term, we may state that it yields the quantity that a given element "represents") yields $Q(\#) = Q(*) = Q(1) = Q(2)$ - all equivalent as unit quantities. However, if we consider 1 and 2 as numbers with their quantitative attributes, then $Q(1) \neq Q(2)$.

This reasoning demonstrates that quantitative attributes are inseparable from the concept of number. Numbers are quantities with associated labels, making bijective correspondences inadequate for defining cardinality of numerical sets. When elements (whether $\#, *$ or 1, 2) are treated as mere labels representing abstract units, the nature of these units becomes indeterminate - they could represent physical objects, empty sets, sets of different sizes, the whole, or entities (e.g., color) lacking quantitative attributes altogether, since we may have distinct objects ($x = o_1$ or $x = o_2$) with different inherent quantitative attributes, such that $Q(o_1) \neq Q(o_2)$, as in the case of two sets of different cardinalities.

An alternative solution exists for defining the quantity of non-negative real numbers in a set while preserving their quantitative and ordinal attributes. Following [9]:

> According to [1] and [11], given a subset U of the real line, we define its diameter as $D(U) = \sup\{|x - y| : x, y \in U\}$ - the greatest distance between any points in U. For a countable collection $\{U_i\}$ of sets with maximum diameter δ covering F (i.e., $F \subseteq \bigcup_{i=1}^{\infty} U_i$ where $0 < D(U_i) \leq \delta$ for each i), we call $\{U_i\}$ a δ-cover of F. We use half-open intervals $[a, b)$ as covering sets to prevent adjacent intervals from intersecting.
>
> For $F \subseteq \mathbb{R}$, define $Q(\delta, F) = \inf \#\{U_i\}$, where $\#\{U_i\}$ counts the covering sets. While unnecessary, we may assume disjointness of the $\{U_i\}$ without affecting $\#\{U_i\}$. Finally, we define the quantity of real numbers in F as:
> $$Q_{\text{Real}}(F) = \lim_{\delta \to 0} Q(\delta, F)$$

This definition preserves the ordered structure of real numbers, treating each number as both a label and a quantity determining its position on the real line. Thus, instead of circularly defining "quantity of quantities," we measure quantities determined by positions in the ordering - effectively counting points on the real line and eliminating definitional circularity.

3 Final Considerations

This work, through its critique of Cantor's bijective correspondence as a method for measuring the quantity of elements in numerical sets, proposes an alternative computational approach for determining this quantity that does not rely on bijective mappings. The proposed method conceptualizes numbers as composite entities consisting of:

- A *label* (nominal identifier)
- An associated *quantity* (a primitive concept)

where the quantitative aspect inherently determines a specific position in the ordering of real numbers, represented by the corresponding point on the real number line.

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